nately, these conditions seem to apply only to a plane wave normally incident upon an infinite flat sheet. Consequently, the point-matching method appears to be a meaningful approximation for nontrivial exterior boundary value problems only if C is nearly circular.

It is certainly easier to write down the equations for the point-matching method than for either the exact integral equation method [11] or the exact extended boundary condition method of Waterman [12], both of which can be based on (2). However, the computational effort required to reach a solution cannot be significantly less for the point-matching method, even though the exact methods involve more work in the setting up of the main computations (a subtle distinction which was pointed out by a reviewer). For equal accuracy (in those instances when the point-matching method is valid) at least as many linear algebraic equations must be solved simultaneously (it is the solution of these equations which absorbs the major part of the computational effort). Consequently, it is suggested that the point-matching method should be discarded in favor of either Harrington's [5] proposed extension of the method or the exact methods [11], [12].

R. H. T. Bates[2]
Dept. of Elec. Engrg.
University of Canterbury
Christchurch, New Zealand

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[2] Formerly with Sperry Rand Research Center, Sudbury, Mass.

Computer-Graphic Analysis of Dielectric Waveguides

The solutions to microwave problems are often enhanced by a visual representation of the fields, especially when the mathematical expressions are so complex as to resist physical interpretation by themselves. The particular graphical aid which is the subject of this correspondence is the field mapping, defined as a family of curves drawn parallel to the vector field being represented. Although such diagrams have been considered to be of great value since the early study of electrodynamics [1], the analytical and numerical complexities of modern engineering problems have inhibited their use on any wide scale. However, the current availability of digital computers, with compatible automatic plotting equipment, has made the numerical determination and display of field mappings a very practical adjunct to established analytical methods. The purpose of this correspondence is to illustrate the utility of field mapping by displaying the transverse electric field for the HE_{11} mode on a dielectric rod. It will be shown that the curvature of the field lines is in the opposite direction to that commonly assumed.

The equipment utilized in this study has been an IBM 7094 computer in conjunction with a Stromberg-Carlson 4020 microfilm unit. The computer performs all the calculations required to construct the field lines and stores the results on magnetic tape. These computed results are then used to control a cathode ray tube display, which is photographed and reproduced by standard methods.

Mathematically, electric field lines in a plane are the solution trajectories of the first-order differential equation

$$\frac{dy}{dx} = \tan\left[\alpha(x, y)\right] = \frac{E_y(x, y)}{E_x(x, y)} \qquad (1)$$

where α is the angle between the electric vector at (x, y) and the $+x$-axis. A first-order numerical approximation to that trajectory passing through a typical boundary point P_0 is depicted in Fig. 1. The calculation is made by using a first-order difference scheme, in which the point (x_{i+1}, y_{i+1}) is determined from (x_i, y_i) according to the relation

$$\begin{aligned} x_{i+1} &= x_i + \delta s \cos \alpha_i \\ y_{i+1} &= y_i + \delta s \sin \alpha_i \end{aligned} \qquad (2)$$

where δs is the path increment. Although this approximation can be refined to whatever accuracy is required [2], it is generally possible to find a path increment sufficiently small to give smooth and accurate contours without impractical amounts of computation.

The application to be considered in this correspondence is the determination of the transverse electric field for the fundamental (HE_{11}) mode of propagation along a dielectric rod with circular cross section. This mode is of significant interest in the analysis of dielectric waveguides [3], dielectric rod antennas [4] and more recently for its application to fiber optics and lasers [5]. Brown and Spector [6] and Snitzer [7] studied the field configuration when the slow wave phase velocity

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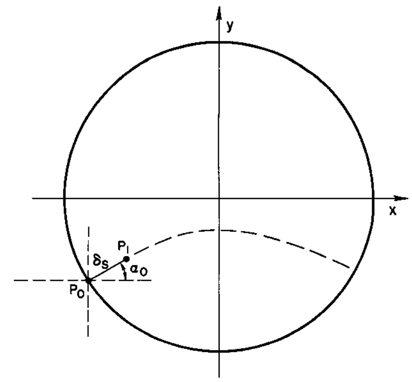

Fig. 1. First-order difference solution for the field lines.

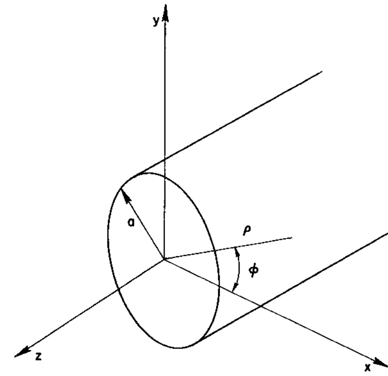

Fig. 2. Cylindrical geometry.

is near the free space velocity (assuming the rod to be in vacuum) and showed that in this limiting situation the field lines in the rod become straight and parallel. Although the more general case of arbitrary dielectric constant has never been accurately studied, it is commonly assumed that the field lines approach the configuration for the TE_{11} mode in a circular waveguide. The apparent justification for this conjecture is that as the wave is slowed down, the power becomes concentrated in the rod in the same way as power is contained within a metallic waveguide. Actually the situation is not analogous, since the fields do not vanish abruptly beyond the surface of a dielectric rod, with the result that there is always some power being transmitted in the region outside.

The cylindrical coordinate representation is shown in Fig. 2. The rod is assumed to have radius a, relative permittivity ϵ_r and a relative permeability of unity. As is well known, the electromagnetic field for the HE_{11} mode is derived from a linear combination of z-directed magnetic and electric Hertz vectors [3]. The electric field may be represented by

$$E = \nabla \times (e_z V_1) + \nabla \times (e_z \times \nabla V_2) \qquad (3)$$

where, for the field inside

$$\begin{aligned} V_1{}^i &= A J_1(\beta\rho) e^{ihz} \begin{cases} \cos\phi \\ \sin\phi \end{cases} \\ V_2{}^i &= B J_1(\beta\rho) e^{ihz} \begin{cases} \sin\phi \\ \cos\phi \end{cases} \end{aligned} \qquad (4)$$

$$\begin{aligned} \beta^2 &= \epsilon_r k_0{}^2 - h^2 \\ k_0 &= \text{free space wavenumber.} \end{aligned} \qquad (5)$$

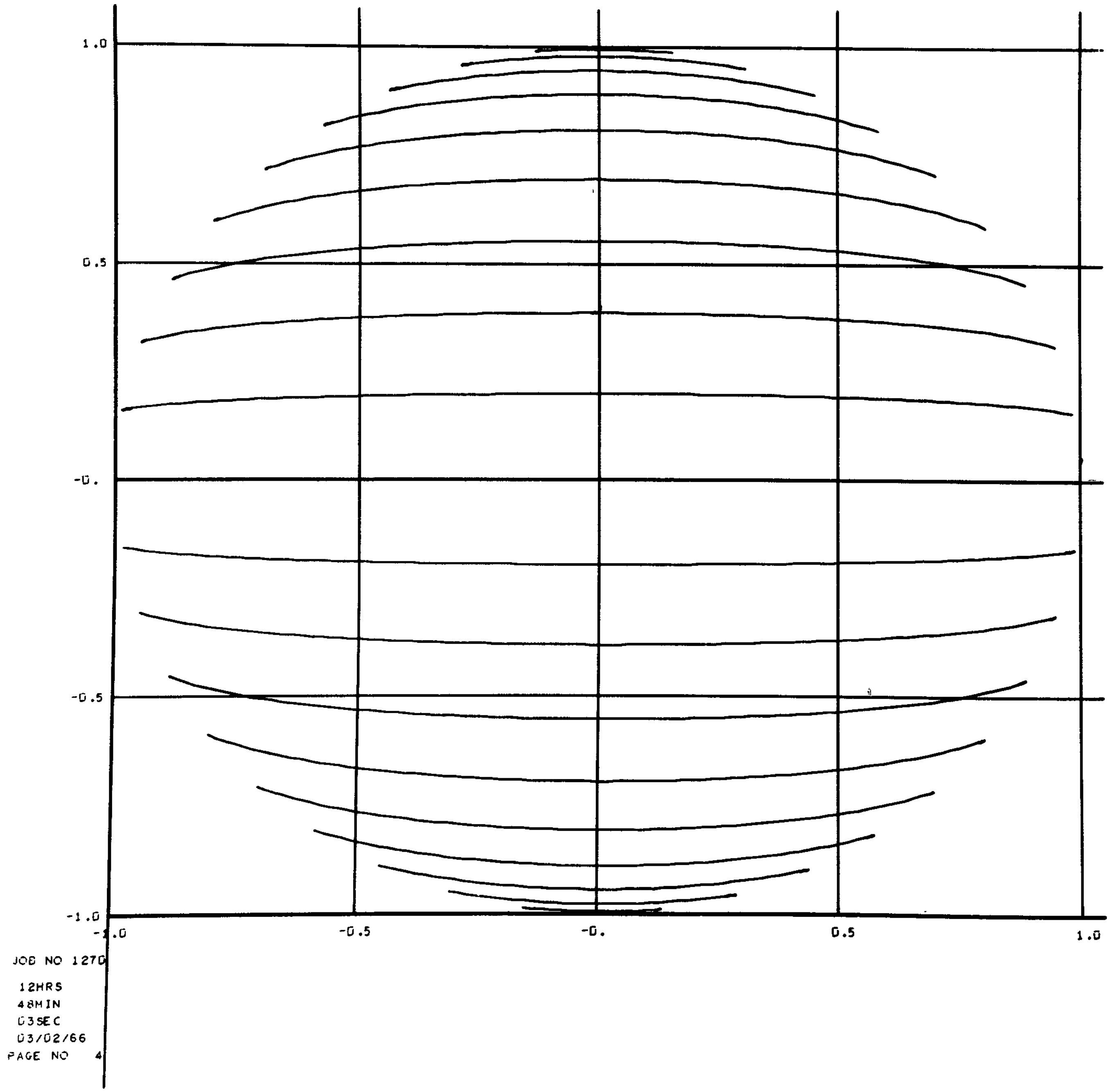

Fig. 3. Computer output, showing transverse electric field inside the rod for the HE_{11} mode. $a/\lambda_0=0.1$, $E_r=12.0$. The coordinates are normalized with respect to the rod radius.

The corresponding potential functions outside are given by

$$V_1^0 = CK_1(\beta_0\rho)e^{ihz}\begin{cases}\cos\phi\\ \sin\phi\end{cases}$$

$$V_2^0 = DK_1(\beta_0\rho)e^{ihz}\begin{cases}\sin\phi\\ \cos\phi\end{cases}$$

$$\beta_0^2 = h^2 - k_0^2 \qquad (6)$$

where $K_1(u)$ denotes the modified Bessel function of the second kind.

The propagation constant h and the relative amplitudes are obtained by requiring that E_ϕ, H_ϕ, E_z, and H_z be continuous at $\rho=a$. The mathematical details are available in Brown and Spector [6].

A typical graphic output is given in Fig. 3, which shows the electric field lines inside a rod with $a/\lambda_0=0.113$ and $\epsilon_r\approx 12.0$, where λ_0 is the free space wavelength. For this case, the wave is slowed to approximately 54 percent of its free space velocity. Notice that the bulging is in the opposite sense to the TE_{11} in a metallic circular guide. Moreover, the field lines inside are not normal to the boundary. They are, however, nearly normal outside, due to the high relative permittivity and the boundary condition for the normal electric field

$$E_\rho^{\text{out}} = \epsilon_r E_\rho^{\text{in}}. \qquad (7)$$

A composite drawing of both the internal and external field configurations, taken from graphical outputs similar to Fig. 3, is shown in Fig. 4.

In this communication we have described a technique and application of numerical mapping of vector field lines, and have found it to be a practical and useful aid in the solution of complex boundary value problems. Other ways in which computer-graphic aids may be utilized are suggested by the availability of motion picture outputs from computers [8]. It is expected that such dynamic representations will be of significant value in such areas as the analysis of elliptically polarized, or nonharmonic electromagnetic fields, where the equations alone are not always sufficient for the results to be fully understood and utilized.

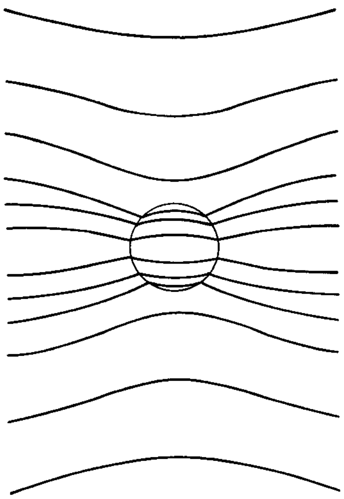

Fig. 4. Complete field mapping for HE_{11} mode. $a/\lambda_0=0.1$, $E_r=12.0$.

Elliott R. Nagelberg
Judith N. Hoffspiegel
Bell Telephone Labs., Inc.
Whippany, N. J.

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On the Reflection of Waves by a Sinusoidally Stratified Half-Space

In this correspondence electromagnetic waves are taken to be obliquely incident on a sinusoidally stratified half-space. Both horizontally and vertically polarized waves are considered, and approximate reflection coefficients are obtained by using some formulas given by Heading [7].

A subject which has received considerable attention recently [1]–[6] is electromagnetic wave propagation in a sinusoidally stratified medium. This topic has application, for example, to the theory of electromagnetic waves in a plasma through which acoustic waves are propagating. In a planar stratified medium, the electromagnetic field can be expressed as the sum of two partial fields which propagate independently. These are often referred to as horizontally and vertically polarized waves, in which the electric and magnetic vectors, respectively, are parallel to the stratifications. Heading [7] studied the reflection of electromagnetic waves in a planar stratified medium by expressing the reflected field as an integral over contributions scattered back from elementary layers of thickness δz situated at level z in the medium. Approximate expressions for the reflection coefficients were obtained by using the Born approximation, thus neglecting multiple scattering. In this correspondence, Heading's single scattering formulas are applied to the problem of reflection from a sinusoidally stratified half-space.

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Suppose that the region $z<0$ is a homogeneous dielectric of permittivity ϵ. The region $z>0$ is taken to have permittivity $\epsilon_r(z)$ relative to that of the medium in $z<0$, where

$$\epsilon_r(z) = \bar{\epsilon}_r[1 - A\cos(2\pi z/d + \phi)]. \quad (1)$$

In this equation $\bar{\epsilon}_r$, A, d, and ϕ are independent of z. The permeability μ is taken to be the same for all z. Losses are neglected so that the permittivity and permeability are everywhere real. Suppose that electromagnetic waves are obliquely incident from the region $z<0$. The configuration is shown in Fig. 1. Rectangular Cartesian coordinates x, y, z are used with the fields independent of y. Let E_x and H_x denote the x-components of electric and magnetic fields, with similar notation for the other components. In horizontally and vertically polarized waves the nonzero components are E_y, H_x, and H_z in the former, and H_y, E_x, and E_z in the latter.

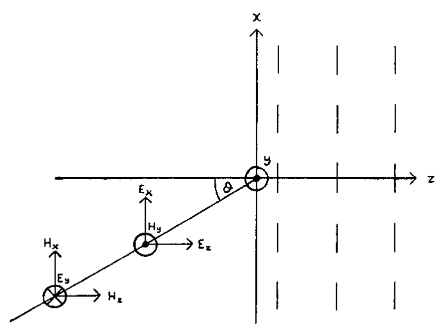

Fig. 1. The configuration.

Let the reflection coefficients for horizontally and vertically polarized incident waves be R_h and R_v, respectively. Heading's single scattering results are

$$R_h \doteqdot \frac{ik}{2C}\int_0^\infty [1 - \epsilon_r(z)]e^{-i2kCz}dz \quad (2)$$

and

$$R_v \doteqdot \frac{ik}{2C}(2C^2 - 1)\int_0^\infty [1 - \epsilon_r(z)] \cdot e^{-i2kCz}dz. \quad (3)$$

These formulas are applicable for a time factor $e^{i\omega t}$, where ω is the angular frequency and t the time. In them $k=\omega(\mu\epsilon)^{1/2}$ and $C=\cos\theta$, θ being the angle of incidence. Equations (2) and (3) are expected to be useful approximations when the resulting reflection coefficients are of small magnitude. They apply in the case of weak scattering so that $\bar{\epsilon}_r$ should be near unity and A should be small.

Equation (2) will now be evaluated, $\epsilon_r(z)$ being given by (1). Use is made of the integral

$$\int e^{ax}\cos(bx + c)dx = \frac{e^{ax}}{a^2 + b^2}[a\cos(bx + c) + b\sin(bx + c)]. \quad (4)$$

In evaluating the integral in (2) at the upper limit, it is assumed that k has a small negative imaginary part which is later allowed to tend to zero. The result is

$$R_h \doteqdot \frac{1 - \bar{\epsilon}_r}{4C^2} + \frac{\bar{\epsilon}_r A}{(kCd/\pi)^2 - 1}\left(\frac{kd}{2\pi}\right)^2 \cdot \left(\cos\phi + \frac{i\pi}{kCd}\sin\phi\right). \quad (5)$$

The term $(1-\bar{\epsilon}_r)/4C^2$ is the approximate reflection coefficient when the medium in $z>0$ is homogeneous and represented by $\bar{\epsilon}_r$. Thus, this term can be regarded as the reflection coefficient for an "averaged medium." The rest of (5) allows for the effects of the modulation of the permittivity about its average value.

Suppose now that $\bar{\epsilon}_r=1$. That is, the average permittivity of the half-space $z>0$ is equal to the permittivity of the half-space $z<0$. If ϕ is now taken to have the values 0 or $\pi/2$, (5) reduces to

$$R_h \doteqdot \frac{A(kd/2\pi)^2}{(kCd/\pi)^2 - 1} \quad (6)$$

and

$$R_h \doteqdot \frac{(iA/2C)(kd/2\pi)}{(kCd/\pi)^2 - 1}, \quad (7)$$

respectively. These are equivalent to (53) and (55) of Tamir et al. [1]. In these two cases, the change in permittivity across the boundary $z=0$ is a maximum (for fixed A) and zero, respectively.

It has been pointed out by a reviewer that the reflection coefficients will not be valid for wavelengths satisfying the Bragg condition. With $\bar{\epsilon}_r$ near unity and A small the Bragg condition is

$$kCd \doteqdot n\pi \qquad n = 1, 2, 3, \cdots. \quad (8)$$

In particular, when the first ($n=1$) Bragg condition is satisfied, (5)–(7) become infinite.

The work of Tamir et al. [1] was restricted to the case of horizontally polarized waves. Then the fields in the sinusoidally stratified medium can be expressed exactly in terms of solutions to Mathieu's equation. Particular attention was paid to the situation in which the permittivity modulations are small [1] and results were obtained by approximating the exact solutions. The case of vertically polarized waves is more complicated; the differential equation governing the fields in the sinusoidally stratified medium is Hill's equation. A series solution was obtained by Yeh et al. [3]. It is of particular interest to note that, for the reflection problem considered here, Heading's formulas give results for both horizontally and vertically polarized waves. A